

The idea of locality

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Abstract

This is a review of recent results on conformal (super)algebras. It may be viewed as an amplification of my Wigner medal acceptance speech (given in July 1996 in Goslar, Germany) reproduced in the introduction.

0 Introduction.

It is a well kept secret that the theory of Kac-Moody algebras has been a disaster.

True, it is a generalization of a very important object, the simple finite-dimensional Lie algebras, but a generalization too straightforward to expect anything interesting from it. True, it is remarkable how far one can go with all these e_i 's, f_i 's and h_i 's. Practically all, even most difficult results of finite-dimensional theory, such as the theory of characters, Schubert calculus and cohomology theory, have been extended to the general set-up of Kac-Moody algebras. But the answer to the most important question is missing: what are these algebras good for? Even the most sophisticated results, like the connections to the theory of quivers, seem to be just scratching the surface.

However, there are two notable exceptions. The best known one is, of course, the theory of affine Kac-Moody algebras. This part of the Kac-Moody theory has deeply penetrated many branches of mathematics and physics. The most important single reason for this success is undoubtedly the isomorphism of affine algebras and central extensions of loop algebras, often called current algebras.

The second notable exception is provided by Borcherds' algebras which are roughly speaking the spaces of physical states of certain chiral algebras.

At this point a natural question arises: what do these notable exceptions have in common? The answer, in my opinion, lies in the idea of locality.

The concept of locality is a beautiful synthesis of three ideas:

1. Einstein's special relativity postulate,
2. Heisenberg's uncertainty principle,
3. the notion of quantum field.

*Partially supported by NSF grant DMS-9622870

As far as I know, the concept of locality was first rigorously formulated by Pauli and then incorporated by Wightman in his axiomatics of quantum field theory.

The locality axiom states that quantum fields, whose supports are space-like separated, must commute.

The locality axiom is empty for 1-dimensional space-time, but becomes a very complicated condition for higher-dimensional space-times. However, in the 2-dimensional case the light cone is a union of two straight lines and one may consider chiral quantum fields, that is, fields depending on one null coordinate.

For chiral fields the locality axiom reduces to a very simple algebraic condition. It just states that the commutator of chiral fields is a finite linear combination, with coefficients being chiral fields, of the delta-function and its derivatives.

Coming back to our two notable exceptions, we see that both are spanned by Fourier coefficients of pairwise local chiral fields!

Now, if we look around, we will immediately find that all the important Lie algebras (or superalgebras) satisfy this locality property: the Virasoro algebra and its marvelous super generalizations, the W -algebras, the Lie algebra $W_{1+\infty}$, etc., etc.

This leads us to the program of study of Lie algebras and superalgebras satisfying the sole locality property. It is an extensive program, but I hope that its main features will be worked out by the year 2000.

The basic message I wanted to convey by my speech is this. Some of the best ideas come to my field from the physicists. And on top of this they award me a medal. One couldn't hope for a better deal.

Thank you.

1 Preliminaries on Lie superalgebras of formal distributions and conformal superalgebras.

A *formal distribution* (usually called a field by physicists) with coefficients in a complex vector space U is a generating series of the form:

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

where $a_{(n)} \in U$ and z is an indeterminate.

Two formal distributions $a(z)$ and $b(z)$ with coefficients in a Lie superalgebra \mathfrak{g} are called (mutually) *local* if for some $N \in \mathbb{Z}_+$ one has:

$$(z - w)^N [a(z), b(w)] = 0 \text{ for some } N \in \mathbb{Z}_+. \quad (1)$$

Introducing the *formal delta-function*

$$\delta(z - w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w} \right)^n,$$

we may write a condition equivalent to (1):

$$[a(z), b(w)] = \sum_{j=0}^{N-1} (a_{(j)}b)(w) \partial_w^j \delta(z-w)/j! \quad (2)$$

for some formal distributions $(a_{(j)}b)(w)$ ([1], Theorem 2.3), which are uniquely determined by the formula

$$(a_{(j)}b)(w) = \text{Res}_z (z-w)^j [a(z), b(w)]. \quad (3)$$

Formula (3) defines a \mathbb{C} -bilinear product $a_{(j)}b$ for each $j \in \mathbb{Z}_+$ on the space of all formal distributions with coefficients in \mathfrak{g} .

Note also that the space (over \mathbb{C}) of all formal distributions with coefficients in \mathfrak{g} is a (left) module over $\mathbb{C}[\partial]$, where the action of $\partial = \partial_z$ is defined in the obvious way, so that $\partial_z a(z) = \sum_n (\partial a)_{(n)} z^{-n-1}$, where $(\partial a)_{(n)} = -n a_{(n-1)}$.

The Lie superalgebra \mathfrak{g} is called a *Lie superalgebra of formal distributions* if there exists a family F of pairwise local formal distributions whose coefficients span \mathfrak{g} . In such a case we say that the family F *spans* \mathfrak{g} . We will write (\mathfrak{g}, F) to emphasize the dependence of F .

The simplest example of a Lie superalgebra of formal distributions is the *current superalgebra* $\tilde{\mathfrak{g}}$ associated to a Lie superalgebra \mathfrak{g} :

$$\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}.$$

It is spanned by the following family of pairwise local formal distributions ($a \in \mathfrak{g}$):

$$a(z) = \sum_{n \in \mathbb{Z}} (t^n \otimes a) z^{-n-1}.$$

Indeed, it is immediate to check that

$$[a(z), b(w)] = [a, b](w) \delta(z-w).$$

The simplest example beyond current algebras is the (centerless) *Virasoro algebra*, the Lie algebra with the basis L_n ($n \in \mathbb{Z}$) and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}.$$

It is spanned by the local formal distribution $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, since one has:

$$[L(z), L(w)] = \partial_w L(w) \delta(z-w) + 2L(w) \partial_w \delta(z-w). \quad (4)$$

Note that $\mathbb{C}[\partial]F$ is a $\mathbb{C}[\partial]$ -submodule of the space of formal distributions which still consists of pairwise local formal distributions. The Lie superalgebra \mathfrak{g} of formal distributions spanned by F is called *simple* if \mathfrak{g} contains no non-trivial ideals spanned by a subspace of $\mathbb{C}[\partial]F$.

The Virasoro algebra has no ideals at all, but the current algebra always has “evaluation” ideals. Nevertheless the current algebra $\tilde{\mathfrak{g}}$ associated to Lie superalgebra \mathfrak{g} is simple in the above sense iff \mathfrak{g} is simple in the usual sense.

Given a Lie superalgebra of formal distributions (\mathfrak{g}, F) , we may always include F in the minimal family F^c of pairwise local formal distributions which is closed under ∂ and all products (3) ([1], § 2.7).

We say that a Lie superalgebra (\mathfrak{g}, F) of formal distribution is *finite* if F^c is a finitely generated $\mathbb{C}[\partial]$ -module.

The Virasoro algebra and the current (super)algebras associated to finite-dimensional Lie (super)algebras are finite. The finiteness condition provides the choice of the “non-twisted” moding, as the following example shows.

Example 1.1 Let \mathfrak{g} be a finite-dimensional Lie algebra and let $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{g}^1$ be a $\mathbb{Z}/2\mathbb{Z}$ -gradation. Then

$$\tilde{\mathfrak{g}} := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}^0 + t^{\frac{1}{2}} \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}^1$$

is a subalgebra of the Lie algebra $\mathbb{C}\left[t^{\frac{1}{2}}, t^{-\frac{1}{2}}\right] \otimes \mathfrak{g}^0$. This is a “twisted” current algebra. It is spanned by the set F of pairwise local formal distributions

$$a(z) = \sum_{n \in \mathbb{Z}} (t^n \otimes a) z^{-n-1} \text{ for } a \in \mathfrak{g}^0, \quad \text{and } a(z) = \sum_{n \in \mathbb{Z}} \left(t^{n+\frac{1}{2}} \otimes a\right) z^{-n-1} \text{ for } a \in \mathfrak{g}^1.$$

However, $[a(z), b(w)] = w[a, b](w)\delta(z-w)$ if $a, b \in \mathfrak{g}^1$. Hence \mathfrak{g} is not a finite Lie algebra of formal distributions, at least if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, since F^c contains all formal distributions $w^n a(w)$, $n \in \mathbb{Z}_+$.

Lie superalgebras of formal distributions can be studied via conformal superalgebras introduced in [1].

A *conformal superalgebra* R is a left $(\mathbb{Z}/2\mathbb{Z}$ -graded) $\mathbb{C}[\partial]$ -module R with a \mathbb{C} -bilinear product $a_{(n)}b$ for each $n \in \mathbb{Z}_+$ such that the following axioms hold ($a, b, c \in R$, $m, n \in \mathbb{Z}_+$):

$$(C0) \quad a_{(n)}b = 0 \text{ for } n \gg 0,$$

$$(C1) \quad (\partial a)_{(n)}b = -na_{(n-1)}b,$$

$$(C2) \quad a_{(n)}b = (-1)^{p(a)p(b)} \sum_{j=0}^{\infty} (-1)^{j+n+1} (\partial^j / j!) b_{(n+j)}a,$$

$$(C3) \quad a_{(m)}(b_{(n)}c) = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c + (-1)^{p(a)p(b)} b_{(n)}(a_{(m)}c).$$

Of course, a conformal algebra coincides with its even part, i.e. $p(a) = 0$ for all $a \in R$ in this case. Note the following consequence of (C1) and (C2):

$$(C1') \quad a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b.$$

It follows that ∂ is a derivation of all products (3).

It is shown in [1], Sec.2.7 that if (\mathfrak{g}, F) is a Lie superalgebra of formal distributions, then F^c is a conformal superalgebra with respect to these products.

Conversely, if $R = \oplus_{i \in I} \mathbb{C}[\partial]a^i$ is a free as $\mathbb{C}[\partial]$ -module conformal superalgebra, we may associate to R a Lie superalgebra of formal distributions $(\mathfrak{g}(R), F)$ with the basis $a_{(m)}^i$ ($i \in I, m \in \mathbb{Z}$) and $F = \left\{ a^i(z) = \sum_n a_{(n)}^i z^{-n-1} \right\}_{i \in I}$ with the bracket (cf. (2)):

$$[a^i(z), a^j(w)] = \sum_{k \in \mathbb{Z}_+} (a_{(k)}^i a^j)(w) \partial_w^k \delta(z-w)/k!, \quad (5)$$

so that $F^c = R$.

It is well-known that a finitely generated $\mathbb{C}[\partial]$ -module R is a direct sum of r copies of $\mathbb{C}[\partial]$ and a d -dimensional (over \mathbb{C}) $\mathbb{C}[\partial]$ -invariant subspace. The number r is called the *rank* of R , and the pair of numbers (r, d) is called the *size* of R . A factor module of R by a non-zero submodule has smaller size (r', d') in the sense that either $r' < r$ or $r' = r$ but $d' < d$. We shall say that R is *finite* if both numbers r and d are finite (which is equivalent to R being a finitely generated $\mathbb{C}[\partial]$ -module).

The *center* of R is the set of the elements $a \in R$ such that $a_{(n)}R = 0$ for all $n \in \mathbb{Z}_+$ (then, by (C2), $R_{(n)}a = 0$ for all $n \in \mathbb{Z}_+$ as well).

It is shown in [1], Proposition 2.7 that a finite (as a $\mathbb{C}[\partial]$ -module) conformal superalgebra R with a trivial center is a free $\mathbb{C}[\partial]$ -module (cf. Lemma 2.1 in §2). Hence the above discussion implies the following proposition.

Proposition 1.1 *Every finite superalgebra of formal distributions (\mathfrak{g}, F) with trivial center is a quotient of a Lie superalgebra $\mathfrak{g}(R)$, where R is a finite conformal superalgebra with trivial center, by an ideal that does not contain all $a_{(n)}$, $n \in \mathbb{Z}$, for a non-zero element $a \in R$. For such Lie superalgebras the $\mathbb{C}[\partial]$ -module F^c is free.*

The simplest example of a finite conformal superalgebra is the *current conformal superalgebra* associated to a finite-dimensional Lie superalgebra \mathfrak{g} :

$$R(\tilde{\mathfrak{g}}) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathfrak{g},$$

which has size $(\dim \mathfrak{g}, 0)$, with the products defined by:

$$a_{(0)}b = [a, b] \quad a_{(j)}b = 0 \text{ for } j > 0, \quad a, b \in \mathfrak{g},$$

and the *Virasoro conformal algebra* $\text{Vir} = \mathbb{C}[\partial]L$, which has size $(1, 0)$, with the products (cf. (4)):

$$L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(j)}L = 0 \text{ for } j > 1.$$

More complicated finite conformal superalgebras are constructed in §5.

Probably the most important infinite conformal algebra is that associated to the Lie algebra \mathcal{D} of differential operators on the circle. Note also that any vertex (=chiral) algebra

[2] may be viewed as a conformal superalgebra if we forget about the products $a_{(n)}b$ for $n < 0$ and let $\partial = T$, the infinitesimal translation operator (see [1], [2]), but this conformal superalgebra is always infinite.

Proposition 1.1 reduces the study of finite superalgebras of formal distributions to that of finite conformal superalgebras. All notions concerning Lie superalgebras are automatically translated into the language of conformal superalgebras and vice versa.

A *subalgebra* S (resp. *ideal* I) of a conformal superalgebra R is a $\mathbb{C}[\partial]$ -submodule of R such that for $a, b \in S$ (resp. $a \in R, b \in I$) we have $a_{(n)}b \in S$ (resp. $\in I$) for all $n \in \mathbb{Z}_+$. Note that the left (or right) ideal is automatically 2-sided due to (C2). The notions of a homomorphism and an isomorphism of conformal superalgebras are obvious. A conformal superalgebra R is called *simple* if its only ideals are 0 and R and not all of the products are trivial.

A Lie superalgebra of formal distributions (\mathfrak{g}, F) with a closed under all products (3) family F is simple iff the associated conformal superalgebra F^c is simple. For example, Vir is a simple conformal algebra and the current algebra $R(\tilde{\mathfrak{g}})$ is a simple conformal superalgebra iff \mathfrak{g} is a simple Lie superalgebra.

The *derived algebra* R' of a conformal superalgebra R is defined as the span over \mathbb{C} of all elements $a_{(j)}b$, where $a, b \in R, j \in \mathbb{Z}_+$. It is easy to see that this is an ideal of R such R/R' is a commutative conformal superalgebra (i.e. all products of R/R' are zero). One defines $R'' = (R')'$, etc, obtaining a descending sequence of ideals $R \supset R' \supset R'' \supset \dots$. The conformal superalgebra R is called *solvable* if n -th member of this sequence is zero for some $n > 0$. The conformal superalgebra R is called *semisimple* if its only solvable ideal is zero.

Define by induction another descending sequence of ideals $R \supset R^1 \supset R^2 \supset \dots$ by letting $R^1 = R', \dots, R^n = \text{span over } \mathbb{C} \text{ of all products } a_{(j)}b \text{ where } a \in R, b \in R^{n-1}, j \in \mathbb{Z}_+$. The conformal superalgebra R is called *nilpotent* if n -th member of this sequence is zero for some $n > 0$.

2 Preliminaries on conformal modules.

Let (\mathfrak{g}, F) be a Lie superalgebra of formal distributions, and let V be a \mathfrak{g} -module. We say that a formal distribution $a(z) \in F$ and a formal distribution $v(z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}$ with coefficients in V are *local* if

$$(z - w)^N a(z)v(w) = 0 \text{ for some } N \in \mathbb{Z}_+. \quad (6)$$

In the same way as in [1], §2.3, one shows that (6) is equivalent to

$$a(z)v(w) = \sum_{j=0}^{N-1} (a_{(j)}v)(w) \partial_w^j \delta(z - w) / j!, \quad (7)$$

for some formal distributions $(a_{(j)}v)(w)$ with coefficients in V , which are uniquely determined by the formula

$$(a_{(j)}v)(w) = \text{Res}_z (z - w)^j a(z)v(w). \quad (8)$$

Example 2.1 Consider the following representation of the (centerless) Virasoro algebra in the vector space V with the basis $v_{(n)}$, $n \in \mathbb{Z}$, over \mathbb{C} :

$$L_m v_{(n)} = ((\Delta - 1)(m + 1) - n) v_{(m+n)} + \alpha v_{(m+n+1)},$$

where $\Delta, \alpha \in \mathbb{C}$. In terms of formal distributions $L(z)$ and $v(z) = \sum_n v_{(n)} z^{-n-1}$ this can be written as follows:

$$L(z)v(w) = (\partial + \alpha)v(w)\delta(z - w) + \Delta v(w)\delta'_w(z - w). \quad (9)$$

Hence $L(z)$ and $v(z)$ are local.

Suppose that V is spanned over \mathbb{C} by coefficients of a family E of formal distributions such that all $a(z) \in F$ are local with all $v(z) \in E$. Then we call (V, E) a *conformal module over (\mathfrak{g}, F)* .

The following is a representation-theoretic analogue (and a generalization) of Dong's lemma (see [1], §3.2).

Lemma 2.1 *Let V be a module over a Lie superalgebra \mathfrak{g} , let $a(z)$ and $b(z)$ (resp. $v(z)$) be formal distributions with coefficients in \mathfrak{g} (resp. V). Suppose that all the pairs (a, b) , (a, v) and (b, v) are local. Then the pairs $(a_{(j)}b, v)$ and $(a, b_{(j)}v)$ are local for all $j \in \mathbb{Z}_+$.*

Proof. We may assume that all three pairs satisfy respectively (1) and (6) for some $N \in \mathbb{Z}_+$. Then we have:

$$\begin{aligned} (z - w)^{3N} (a_{(j)}b)(z)v(w) \\ = (z - w)^N \text{Res}_u \sum_{i=0}^{2N} \binom{2N}{i} (z - u)^i (u - w)^{2N-i} (u - z)^j [a(u), b(z)]v(w). \end{aligned}$$

The summation over i in the right-hand side from 0 to $2N$ may be replaced by that from 0 to N since $a(u)$ and $b(z)$ are mutually local. Hence it can be written as follows:

$$(z - w)^N \text{Res}_u (u - w)^N P(z, u, w) (u - z)^j (a(u)b(z)v(w) - b(z)a(u)v(w))$$

for some polynomial P . But this is zero since both pairs (b, v) and (a, v) are local, which proves that the pair $(b_{(j)}a, v)$ is local.

Next, using the first part of lemma, we may find N for which all pairs $(a_{(j)}b, v)$ and (a, v) satisfy (6). Then we have:

$$\begin{aligned} a(z)(b_{(j)}v)(w) &= \text{Res}_u a(z)b(u)v(w)(u - w)^j \\ &= -\text{Res}_u ([b(u), a(z)]v(w) + b(u)a(z)v(w))(u - w)^j \\ &= -\text{Res}_u \left(\sum_{i \geq 0} (b_{(i)}a)(z)v(w) \partial_z^i \delta(u - z)/i! + b(u)a(z)v(w) \right) (u - w)^j. \end{aligned}$$

Hence $(z - w)^N a(z)(b_{(j)}v)(w) = 0$. \square

This lemma shows that the family E of a conformal module (V, E) over (\mathfrak{g}, F) can always be included in a larger family E^c which is local with respect to F^c and such that $\partial E^c \subset E^c$ and $a_{(j)}E^c \subset E^c$ for all $a \in F$ and $j \in \mathbb{Z}_+$.

It is straightforward to check the following properties for $a, b \in F$ and $v \in E^c$:

$$[a_{(m)}, b_{(n)}]v = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)}v, \quad (10)$$

$$(\partial a)_{(n)}v = [\partial, a_{(n)}]v = -na_{(n-1)}v. \quad (11)$$

(Here $[\cdot, \cdot]$ is the bracket of operators on E^c .) It follows from (10) (by induction on m) and (11), that $a_{(j)}E^c \subset E^c$ for all $a \in F^c$ and $j \in \mathbb{Z}_+$.

Thus, any conformal module (V, E) over a Lie superalgebra of formal distributions (\mathfrak{g}, F) gives rise to a module $M = E^c$ over the conformal algebra $R = F^c$, defined as follows. It is a (left) $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\partial]$ -module with \mathbb{C} -linear maps $a \mapsto a_{(n)}^M$ of R to $\text{End}_{\mathbb{C}}M$ given for each $n \in \mathbb{Z}_+$ such that the following properties hold (cf. (10) and (11)) for $a, b \in R$, $m, n \in \mathbb{Z}_+$:

$$\text{(M0)} \quad a_{(n)}^M v = 0 \text{ for } v \in M \text{ and } n \gg 0,$$

$$\text{(M1)} \quad [a_{(m)}^M, b_{(n)}^M] = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)_{(m+n-j)}^M,$$

$$\text{(M2)} \quad (\partial a)_{(n)}^M = [\partial, a_{(n)}^M] = -na_{(n-1)}^M.$$

Conversely, suppose that a conformal superalgebra R is a free $\mathbb{C}[\partial]$ -module and consider the associated Lie superalgebra of formal distributions $(\mathfrak{g}(R), F)$ (see §1). Let M be a module over the conformal superalgebra R and suppose that M is a free $\mathbb{C}[\partial]$ -module with basis $\{v^\alpha\}_{\alpha \in J}$. This gives rise to a conformal $\mathfrak{g}(R)$ -module $V(M)$ with basis $v_{(n)}^\alpha$, where $\alpha \in J$, $n \in \mathbb{Z}$, defined by (cf. (7)):

$$a^\alpha(z)v^\beta(w) = \sum_{j \in \mathbb{Z}_+} (a_{(j)}^\alpha v^\beta)(w) \partial_w^j \delta(z - w)/j!. \quad (12)$$

Remark 2.1 Given a module M over a conformal superalgebra R , we may change its structure as a $\mathbb{C}[\partial]$ -module replacing ∂ by $\partial + A$ where A is an endomorphism over \mathbb{C} of M which commutes with all $a_{(n)}^M$ (this will not affect axiom (M2)). Sometimes we shall not distinguish these R -modules. For example, we may put $\alpha = 0$ in (9).

A conformal module (V, E) (resp. module M) over a Lie superalgebra of formal distributions (\mathfrak{g}, F) (resp. over a conformal superalgebra R) is called *finite* if E^c (resp. M) is a finite $\mathbb{C}[\partial]$ -module.

Note that the maps $a \mapsto a_{(n)}$ of R to $\text{End}_{\mathbb{C}}R$ define a R -module, called the *adjoint module*. (It is finite iff R is finite).

The following lemma generalizes Proposition 2.7 from [1].

Lemma 2.2 *Let M be a R -module.*

- a) *If $\partial v = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v \in M$, then $R_{(n)}^M v = 0$ for all $n \in \mathbb{Z}_+$.*
- b) *If M is a finite module which has no non-zero invariants, i.e. vectors v such that $R_{(n)}^M v = 0$ for all $n \in \mathbb{Z}_+$, then it is a free $\mathbb{C}[\partial]$ -module.*

Proof. Let $a \in R$ and take the minimal m such that $a_{(j)}v = 0$ for $j \geq m$. We have: $0 = (\partial - \lambda)(a_{(m)}v) = [\partial - \lambda, a_{(m)}]v + a_{(m)}(\partial - \lambda)v = -ma_{(m-1)}v$. This shows that $m = 0$, proving (a).

Since (a) says that a R -module M with only zero invariants has no torsion, (b) follows from (a). \square

The correspondence between finite conformal modules with no non-trivial invariants over a finite Lie superalgebra of formal distributions (\mathfrak{g}, F) with a trivial center and finite conformal modules with no non-trivial invariants over the conformal superalgebra F^c is described in a fashion similar to Proposition 1.1.

Note that Example 2.1 gives a 2-parameter family of (irreducible) modules of size (1,0) over the Virasoro conformal algebra. Note also that the well-known family of graded Vir-modules given by

$$L_m v_{(n)} = ((\Delta - 1)m - n + \alpha)v_{(m+n)}$$

is conformal, but is finite iff $\alpha = \Delta - 1$.

The following simple observation is fundamental for representation theory of conformal superalgebras.

Proposition 2.1 *Consider the Lie superalgebra of formal distributions $(\mathfrak{g}(R), F)$ defined by (5) and let $\mathfrak{g}(R)_+$ be the subalgebra of $\mathfrak{g}(R)$ spanned by all $a_{(n)}^i$ with $i \in I$, $n \in \mathbb{Z}_+$. Denote by $\mathfrak{g}(R)^+$ the semidirect product of the 1-dimensional Lie algebra $\mathbb{C}\partial$ and ideal $\mathfrak{g}(R)_+$ with the action of ∂ on $\mathfrak{g}(R)_+$ given by $\partial(a_{(n)}^i) = -na_{(n-1)}^i$. Then any module M over the conformal superalgebra R gives rise to a $\mathfrak{g}(R)^+$ -module M (over \mathbb{C}) such that*

$$a_{(n)}^i v = 0 \text{ for } v \in M \text{ and } n \gg 0. \quad (13)$$

Corollary 2.1 *Let $R = \bigoplus_{\alpha \in I} \mathbb{C}[\partial]a^\alpha$ be a conformal superalgebra and $M = \bigoplus_{\beta \in J} \mathbb{C}[\partial]v^\beta$ be a free $\mathbb{C}[\partial]$ -module. Then, given $a_{(n)}^\alpha v^\beta \in M$ for all $\alpha \in I$, $\beta \in J$, $n \in \mathbb{Z}_+$, which is 0 for $n \gg 0$, we may extend uniquely the action of $a_{(n)}^\alpha$ to the whole R on M using (M2). Suppose that (M1) holds for all $a = a_{(m)}^\alpha$, $b = a_{(n)}^\beta$. Then M is a R -module.*

Using Proposition 2.1 and Corollary 2.1, one can construct large families of finite modules over conformal superalgebras.

Example 2.2 Let $\text{Vir}_{\geq 0} = \sum_{j \geq 0} \mathbb{C}L_j$ and consider a representation π of $\text{Vir}_{\geq 0}$ in a finite-dimensional (over \mathbb{C}) vector space U . Let A be an endomorphism of U commuting with all

$\pi(L_j)$ ($j \in \mathbb{Z}_+$). Then $\mathbb{C}[\partial] \otimes U$ is a finite module over the conformal algebra Vir defined by the following formulas ($u \in U$):

$$L_{(0)}u = (\partial + A)u, \quad L_{(j)}u = \pi(L_{j-1})u \text{ for } j \geq 1.$$

For example, we can take $\pi(L_0) = B$, where B is an endomorphism of U commuting with A . Then

$$L_{(0)}u = (\partial + A)u, \quad L_{(1)}u = Bu, \quad L_{(j)}u = 0, \text{ for } j > 1,$$

defines a finite module over Vir , which we denote by $M(A, B)$. Taking $\dim U = 1$, $A = \alpha$ and $B = \Delta$ gives Example 2.1.

3 Structure theory of finite conformal algebras.

Results stated in this section is a joint work [3] with my student, Alessandro D'Andrea. The reader is referred to [3] for details.

Theorem 3.1 (conformal Engel theorem) *Let M be a finite module over a finite conformal (super)algebra R such that $a_{(n)}^M$ is a nilpotent operator for all $a \in R$ and $n \in \mathbb{Z}_+$. Then there exists a non-zero vector $v \in M$ such that*

$$a_{(n)}^M v = 0 \text{ for all } a \in R, n \in \mathbb{Z}_+. \quad (14)$$

Corollary 3.1 *A finite conformal (super)algebra R is nilpotent iff $a_{(n)}$ is a nilpotent operator (on R) for all $a \in R$ and $n \in \mathbb{Z}_+$.*

Theorem 3.2 (conformal Lie theorem) *Let M be a finite module over a finite solvable conformal algebra R . Then there exists a non-zero vector $v \in M$ such that*

$$a_{(n)}^M v = \lambda(a, n)v, \text{ where } \lambda(a, n) \in \mathbb{C}, \text{ for all } a \in R, n \in \mathbb{Z}_+. \quad (15)$$

Corollary 3.2 *The derived algebra of a finite solvable conformal algebra is nilpotent.*

Theorem 3.3 *A simple finite conformal algebra is isomorphic either to a current conformal algebra $R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is a simple finite-dimensional Lie algebra, or to the Virasoro conformal algebra Vir .*

It is not quite true that a semisimple finite conformal algebra is a direct sum of simple ones. For example, if \mathfrak{g} is a semisimple finite-dimensional Lie algebra, the semi-direct sum $R = \text{Vir} + R(\tilde{\mathfrak{g}})$ is semisimple, where the products are given by:

$$\begin{aligned} L_{(0)}a &= \partial a \text{ if } a \in R; & L_{(1)}L &= 2L; & L_{(1)}a &= a \text{ if } a \in \mathfrak{g}; \\ a_{(0)}b &= [a, b]; & \text{all other products (up to the order) on } L + \mathfrak{g} & \text{ are 0.} \end{aligned}$$

Theorem 3.4 *A semi-simple finite conformal algebra is isomorphic to the direct sum of conformal algebras of three types:*

- a) *current conformal algebra $R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is simple,*
- b) *conformal algebra Vir ,*
- c) *conformal algebra $\text{Vir} + R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is semisimple.*

Theorem 3.5 *A finite conformal algebra R admits a faithful finite irreducible module iff R is of one of the following types:*

- a) *current conformal algebra $R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is semisimple,*
- b) *current conformal algebra $R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is semisimple or zero plus 1-dimensional,*
- c) *conformal algebra $\text{Vir} + R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is semisimple or zero,*
- d) *conformal algebra $\text{Vir} + R(\tilde{\mathfrak{g}})$, where \mathfrak{g} is semisimple or zero plus 1-dimensional,*
- e) *one of a) or c) plus 1-dimensional (over \mathbb{C}) conformal algebra.*

4 Finite modules over semisimple finite conformal algebras.

Results of this section is a joint work [4] with Sun-Jen Cheng.

The following is a key lemma.

Lemma 4.1 *Let \mathcal{L} be a Lie superalgebra (over \mathbb{C}) with a distinguished element ∂ and a descending sequence of subspaces $\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$ such that $[\partial, \mathcal{L}_n] = \mathcal{L}_{n-1}$ for all $n > 0$. Let V be a \mathcal{L} -module and let*

$$V_n = \{v \in V \mid \mathcal{L}_n v = 0\}, n \in \mathbb{Z}_+.$$

Suppose that $V_n \neq 0$ for $n \gg 0$. Suppose that the minimal $N \in \mathbb{Z}_+$ for which $V_N \neq 0$ is positive. Let v_1, v_2, \dots be a linearly independent over \mathbb{C} set of vectors of V_N which generate $\mathbb{C}[\partial]V_N$ as a $\mathbb{C}[\partial]$ -module. Then v_1, v_2, \dots is a basis (over \mathbb{C}) of V_N and a free set of generators of the $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial]V_N$. In particular, V_N is finite-dimensional over \mathbb{C} if V is a finitely generated $\mathbb{C}[\partial]$ -module.

It is clear from the very definition that the Lie superalgebra $\mathcal{L} = \mathfrak{g}(R)^+$ and any finite conformal module V over R viewed as an \mathcal{L} -module (see Proposition 1.1) satisfy the conditions of Lemma 4.1.

Theorem 4.1 *Any irreducible finite Vir -module is either 1-dimensional over \mathbb{C} , or else is a free of rank 1 $\mathbb{C}[\partial]$ -module $\mathbb{C}[\partial]v$ defined by (cf. Example 2.1):*

$$L_{(0)}v = (\partial + \alpha)v, \quad L_{(1)}v = \Delta v, \quad L_{(j)}v = 0 \text{ for } j > 1$$

for some $\alpha, \Delta \in \mathbb{C}$, $\Delta \neq 0$.

Let \mathfrak{g} be a semisimple finite-dimensional Lie algebra and let $R(\tilde{\mathfrak{g}})$ be the associated current conformal algebra.

Theorem 4.2 *Any irreducible finite $R(\tilde{\mathfrak{g}})$ -module is either 1-dimensional over \mathbb{C} , or else is of the form $\mathbb{C}[\partial] \otimes_{\mathbb{C}} U$, where U is a non-trivial irreducible finite-dimensional \mathfrak{g} -module, with the action of $R(\tilde{\mathfrak{g}})$ defined by:*

$$a_{(j)}u = 0 \text{ for } j > 0, \quad a_{(0)}u = au, \text{ if } a \in \mathfrak{g}, \quad u \in U.$$

Theorem 4.3 *Any irreducible finite $\text{Vir} + R(\tilde{\mathfrak{g}})$ -module is either the trivial 1-dimensional over \mathbb{C} , or else is of the form $\mathbb{C}[\partial] \otimes_{\mathbb{C}} U$, where U is a non-trivial irreducible finite-dimensional \mathfrak{g} -module, with the action of $\text{Vir} + R(\tilde{\mathfrak{g}})$ defined by:*

$$\begin{aligned} a_{(j)}u &= \delta_{j0}au \text{ if } a \in \mathfrak{g}, \quad u \in U, \quad j \in \mathbb{Z}_+, \\ L_{(0)}u &= \partial u, \quad L_{(j)}u = \delta_{1j}\Delta u \text{ if } u \in U, \quad j \geq 1, \text{ where } \Delta \in \mathbb{C}. \end{aligned}$$

Thus, we see that the classification of irreducible finite modules over semi-simple conformal algebras is similar to that of semi-simple Lie algebra, a small difference being caused by the Virasoro conformal algebra.

There is, however, an essentially new feature: complete reducibility breaks down in a very dramatic way. For example, the module $M(A, B)$ over Vir constructed in Example 2.2 is indecomposable iff the pair of commuting operators (A, B) is indecomposable. Similar examples may be constructed for current conformal algebras as well.

We describe below all the extensions between an irreducible Vir -module $M(\alpha, \Delta)$, $\alpha, \Delta \in \mathbb{C}$, $\Delta \neq 0$, and the trivial Vir -module \mathbb{C} .

Theorem 4.4 (a) *A non-split extension of Vir -modules of the form*

$$0 \longrightarrow M(\alpha, \Delta) \longrightarrow V \longrightarrow \mathbb{C} \longrightarrow 0$$

exists iff $\Delta = 1$ which is

$$0 \longrightarrow M(\alpha, 1) \longrightarrow M(\alpha, 0) \longrightarrow \mathbb{C} \longrightarrow 0.$$

(b) *A non-split extension of Vir -modules of the form*

$$0 \longrightarrow \mathbb{C} \longrightarrow V \longrightarrow M(\alpha, \Delta) \longrightarrow 0$$

exists iff $\Delta = 1$ or 2 . In these two cases $V = \mathbb{C}[\partial]v + \mathbb{C}$, where \mathbb{C} is the trivial submodule, with the action on v :

$$L_{(0)}v = (\partial + \alpha)v, \quad L_{(1)}v = \Delta v, \quad L_{(j)}v = \delta_{j, \Delta+1}1 \in \mathbb{C} \text{ for } j > 1.$$

(c) [18] *A non-split extension of Vir -modules of the form*

$$0 \longrightarrow M(\alpha, \Delta) \longrightarrow V \longrightarrow M(\beta, \Delta') \longrightarrow 0$$

may exist only when $\alpha = \beta$ and $\Delta' - \Delta = 0, 1, 2, 3, 4, 5, 6$. They can be explicitly described using the results of [5].

Denote by $M(\lambda)$ the $R(\tilde{\mathfrak{g}})$ -module $\mathbb{C}[\partial] \otimes F(\lambda)$, where $F(\lambda)$ is the finite-dimensional irreducible highest weight module over the simple Lie algebra \mathfrak{g} (see Theorem 4.2).

Theorem 4.5 *The only non-split extension between 1-dimensional $R(\tilde{\mathfrak{g}})$ -module and a module $M(\lambda)$ is of the form:*

$$0 \longrightarrow \mathbb{C} \longrightarrow V \longrightarrow M(\theta) \longrightarrow 0,$$

where θ is the highest root (so that $F_\theta = \mathfrak{g}$). In this case $V = \mathbb{C}[\partial] \otimes \mathfrak{g} + \mathbb{C}$, where \mathbb{C} is a submodule, with the action on \mathfrak{g} :

$$a_{(0)}b = [a, b], \quad a_{(j)}b = \delta_{1j}(a \mid b) \in \mathbb{C} \text{ for } j \geq 1,$$

where (\mid) is the Killing form on \mathfrak{g} .

5 Classification of simple finite conformal superalgebras.

The list of all finite conformal superalgebras is much richer than that of finite conformal algebras. First, there are many more simple finite-dimensional Lie superalgebras (classified in [6]), and the associated conformal superalgebra is finite and simple. Second, there are many “superizations” of the Virasoro conformal algebra. We describe them below. They are associated to superconformal algebras constructed in [6], [7] and [8] (cf. [9]).

Let $\Lambda(N)$ denote the Grassmann algebra over \mathbb{C} in N indeterminates ξ_1, \dots, ξ_N . Let $W(N)$ be the Lie superalgebra of all derivations of superalgebra $\Lambda(N)$. It consists of all linear differential operators $\sum_{i=1}^N P_i \partial_i$, where $P_i \in \Lambda(N)$ and ∂_i stands for the partial derivative by ξ_i .

The first series of examples is the series conformal superalgebras W_N of rank $(N + 1)2^N$:

$$W_N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} (W(N) \oplus \Lambda(N))$$

with the following products $(a, b \in W(N), f, g \in \Lambda(N))$:

$$\begin{aligned} a_{(j)}b &= \delta_{j0}[a, b], \quad a_{(0)}f = a(f), \quad a_{(j)}f = -\delta_{j1}(-1)^{p(a)p(f)}fa \text{ if } j \geq 1, \\ f_{(0)}g &= -\partial(fg), \quad f_{(j)}g = -2\delta_{j1}fg \text{ if } j \geq 1. \end{aligned}$$

For an element $D = \sum_{i=1}^N P_i(\partial, \xi) \partial_i + f(\partial, \xi) \in W_N$ define *divergence* by the formula

$$\text{div } D = \sum_{i=1}^N (-1)^{p(P_i)} \partial_i P_i + \partial f.$$

The second series of examples is

$$S_N = \{D \in W_N \mid \text{div } D = 0\}.$$

This is a conformal superalgebra of rank $N2^N$, and it is simple iff $N \geq 2$.

The third series of examples is K_N . It is also a subalgebra of W_N (of rank 2^N), but it is more convenient to describe it as follows:

$$K_N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \Lambda(N)$$

with the following products ($f, g \in \Lambda(N)$):

$$\begin{aligned} f_{(0)}g &= \left(\frac{1}{2}|f| - 1 \right) \partial fg + \frac{1}{2}(-1)^{|f|} \sum_{i=1}^N (\partial_i f)(\partial_i g), \\ f_{(j)}g &= \left(\frac{1}{2}(|f| + |g|) - 2 \right) \delta_{j1} fg \text{ if } j \geq 1. \end{aligned}$$

We assume here that f and g are homogeneous elements of degrees $|f|$ and $|g|$ in the gradation defined by $\deg \xi_i = 1$ for all i .

These three series include all well-known examples. Thus, $W_0 \simeq K_0$ is the Virasoro conformal algebra, K_1 is the Neveu-Schwarz conformal superalgebra, $K_2 \simeq W_1$ and K_3 are the $N = 2$ and 3 conformal superalgebras, S_2 is the $N = 4$ conformal superalgebra. Nevertheless there is one exceptional example constructed in [8]. It is the subalgebra CK_6 of rank 32 in the conformal superalgebra K_6 spanned over $\mathbb{C}[\partial]$ by the following elements:

$$1 + \alpha \partial^3 \nu, \quad \xi_i - \alpha \partial^2 \xi_i^*, \quad \xi_i \xi_j + \alpha \partial(\xi_i \xi_j)^*, \quad \xi_i \xi_j \xi_k + \alpha(\xi_i \xi_j \xi_k)^*.$$

Here $\alpha^2 = -1$, $\nu = \xi_1 \xi_2 \dots \xi_6$, $(\xi_{i_1} \xi_{i_2} \dots)^* = \partial_{i_1} \partial_{i_2} \dots \nu$.

Remark 5.1 Let $\mathfrak{g}'_1 = \mathfrak{g}(CK_6)$ (resp. $\mathfrak{g} = \mathfrak{g}(K_6)$) be the Lie superalgebra associated to the conformal superalgebra CK_6 (resp. K_6) and let \mathfrak{g}'_+ (resp. \mathfrak{g}_+) be its subalgebra spanned by all non-negative modes of all formal distributions of CK_6 (resp. K_6). The Lie superalgebra \mathfrak{g}_+ has a \mathbb{Z} -gradation of the form $\bigoplus_{j \geq -2} \mathfrak{g}_j$ with $\mathfrak{g}_{-2} = \mathbb{C}$, $\mathfrak{g}_{-1} = \mathbb{C}^6$, $\mathfrak{g}_0 = so_6 \oplus \mathbb{C}$, $\mathfrak{g}_1 = \bigwedge^3 \mathbb{C}^6 \oplus \mathbb{C}^6$ [6]. The reason for occurrence of CK_6 lies in the fact that the so_6 -module $\bigwedge^3 \mathbb{C}^6 \simeq V \oplus V^*$ is reducible, where V is a 10-dimensional so_6 -module. The subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus (V + \mathbb{C}^6)$ generates the simple \mathbb{Z} -graded superalgebra \mathfrak{g}'_+ , which should be added to the list of Theorem 10 from [6].

Theorem 5.1 *Any simple finite conformal superalgebra is isomorphic either to the current conformal superalgebra associated to a simple finite-dimensional Lie superalgebra (classified in [6]), or to one of the following conformal superalgebras ($N \in \mathbb{Z}_+$):*

$$W_N, \quad S_{N+2}, \quad K_N, \quad CK_6.$$

The proof of this theorem is pretty long [10]. It follows the same strategy as [6], the first step have been worked out in [9].

6 Work in progress.

6.1 Cohomology [11]

In representation theory and conformal field theory it is important to consider central extensions of Lie (super)algebras of formal distributions. In the language of a conformal (super)algebra R it amounts to considering a conformal superalgebra

$$\tilde{R} = R \oplus \mathbb{C}C \text{ with } p(C) = 0, \quad \partial C = 0, \quad C_{(n)}\tilde{R} = 0 \text{ for } n \in \mathbb{Z}_+.$$

The n -th product $a_{(\tilde{n})}b$ on $R \subset \tilde{R}$ is given by

$$a_{(\tilde{n})}b = a_{(n)}b + \alpha_n(a, b)C.$$

It is straightforward to see that \tilde{R} is a conformal superalgebra iff the sequence $\{\alpha_n\}_{n \in \mathbb{Z}_+}$ is a 2-cocycle on R defined as follows. It is a sequence of \mathbb{C} -valued \mathbb{C} -bilinear forms $\alpha_n (n \in \mathbb{Z}_+)$ on $R \times R$ such that $(a, b, c \in R, m, n \in \mathbb{Z}_+)$:

$$\begin{aligned} \alpha_n(\partial a, b) &= -n\alpha_{n-1}(a, b), \\ \alpha_n(a, b) &= (-1)^{n+1+p(a)p(b)}\alpha_n(b, a), \\ \alpha_m(a, b_{(n)}c) &= \sum_{j=0}^m \binom{m}{j} \alpha_{m+n-j}(a_{(j)}b, c) + (-1)^{p(a)p(b)}\alpha_n(b, a_{(m)}c). \end{aligned}$$

As usual, the trivial cocycle $\alpha_n(a, b) = f(a_{(n)}b)$, where $f : R \rightarrow \mathbb{C}$ is a \mathbb{C} -linear map, defines a trivial central extension of \mathfrak{g} (isomorphic to the direct sum of \mathfrak{g} and \mathbb{C}). Two cocycles that differ by a trivial cocycle are called equivalent; they define isomorphic central extensions.

Starting with this observation, Alexander Voronov and I developed in [11] a cohomology theory of conformal algebras. This cohomology $H^*(R, M)$ is defined for any R -module M in a way similar to the Lie algebra cohomology. The central extensions of R are parameterized by $H^2(R, \mathbb{C})$.

Examples.

- a) $\dim_{\mathbb{C}} H^2(R(\tilde{\mathfrak{g}}), \mathbb{C}) = 1$ if \mathfrak{g} is a simple finite-dimensional Lie algebra. The non-trivial 2-cocycle is given by

$$\alpha_n(a, b) = \delta_{n,1}(a | b) \text{ for } a, b \in \mathfrak{g}.$$

- b) [7], [8]. Let R be one of the simple conformal superalgebras W_N, S_N, K_N, CK_6 . Then $\dim_{\mathbb{C}} H^2(R, \mathbb{C}) = 0$ or 1 and the latter case takes place iff $R = W_N$ with $N \leq 2, S_2$, and K_N with $N \leq 4$.
- c) [11] $\dim_{\mathbb{C}} H^n(\text{Vir}, \mathbb{C}) = 1$ if $n = 0, 2$ or 3, and $= 0$ otherwise. This example is intimately related to the Gelfand-Fuchs cohomology [5].

6.2 Modules over finite simple conformal superalgebras [12]

Combining methods described in §4 with the ones of [13] Alexey Rudakov and I were able to classify and construct all finite irreducible modules over all finite simple conformal superalgebras. For current conformal superalgebras and for K_1 (Neveu-Schwarz) it was done already in [4], and the answer is similar to Theorem 4.1 and 4.2 respectively. However, in all other cases some interesting new effects occur.

6.3 Finite modules over infinite conformal algebras [15]

The next after finite conformal (super)algebras is the class of conformal algebras R that admit a faithful finite module M . Suppose for simplicity that M is a free module of rank N over $\mathbb{C}[\partial]$. It turns out that R is then a subalgebra of the “general” conformal algebra gc_N of all “conformal” endomorphisms of M . A conformal endomorphism A of M is a sequence of endomorphisms $A_{(n)}$ of M over \mathbb{C} for each $n \in \mathbb{Z}_+$ such that $A_{(n)}v = 0$ for $n \gg 0$ and $A_{(n)}\partial v = \partial A_{(n)}v + nA_{(n-1)}v, v \in M$. All conformal endomorphisms of M form an infinite conformal algebra, denoted by gc_N , with $\mathbb{C}[\partial]$ -module structure $(\partial A)_{(n)} = -nA_{(n-1)}$ and the following products for each $m \in \mathbb{Z}_+$:

$$(A_{(m)}B)_{(n)} = \sum_{j=0}^m (-1)^{m+j} \binom{m}{j} [A_{(j)}, B_{(m+n-j)}],$$

M being its finite irreducible module. This conformal algebra is simple, but infinite.

It is easy to see that gc_N is the conformal algebra associated to the Lie algebra of $N \times N$ matrix differential operators on the circle. Using the methods of [14], we show in [15] that a kind of an analogue of the Burnside theorem holds: gc_N has a unique non-trivial finite irreducible module (up to going to the contragredient module) and any finite module is a direct sum of irreducible modules.

It is a challenging problem to classify all (infinite) simple conformal algebras that have a finite module. As we have seen, all of them are subalgebras of gc_N , some N . Incidentally, it follows that they have finite growth, and it is natural to state a more general problem: classify all simple conformal algebras of finite growth.

Note that vertex algebras viewed as conformal algebras have an exponential growth.

6.4 q -analogues [16]

One can develop a parallel theory of q -analogues of conformal algebras. They include the “twisted” current conformal algebras. The general q -conformal algebra is that associated with the *sin* algebra [17].

We hope that this will lead eventually to q -analogues of vertex algebras.

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